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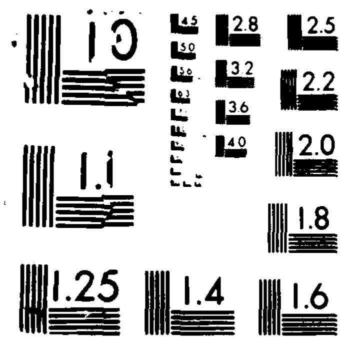
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This article contains two results relating to the coverage of a circle by random arcs having random sizes. The results extend and complement the results of Siegel (1978) and Siegel and Holst (1982). Other papers which deal with random arcs of random sizes are those of Jewell and Romano (1982), Yadin and Zacks (1982), Janson (1983) and Huffer (1986).

Consider n arcs placed uniformly and independently on a circle with circumference 1. The arc lengths are independent and identically distributed according to the distribution F on $(0,1)$. The vacancy is that part of the circle which is not covered by any arc. Let V denote the total length of the vacancy. G will denote the number of uncovered gaps on the circle. The vacancy consists of G disjoint segments.

The first result concerns the joint moments of V and G. These joint moments are shown to satisfy

$$(1) \qquad EV^p\binom{G}{q} = \binom{n}{q} E\left\{\left(\prod_{i=1}^{q} F(S_i)\right)\left(\sum_{j=1}^{p+q}\phi(S_j)\right)^{n-q}\right\}.$$

Here $\binom{G}{q}$ is the ordinary binomial coefficient (which is defined to be zero if $G < q$) and $\phi(s) = \int_0^s F(x)dx$. The random variables $S_1, S_2, \cdots, S_{p+q}$ are the lengths of the spaces between $p+q$ points chosen uniformly and independently on the circle. The vector $(S_1, S_2, \cdots, S_{p+q})$ is uniformly distributed on the simplex determined by $\Sigma_i S_i = 1$.

The second result concerns the distribution of V in the special case where $F(x) = x^\beta$ for some $\beta > 0$. The conditional distribution of V given G is shown to be a Beta distribution. More precisely,

$$(2) \qquad \mathcal{L}(V \mid G = k) = \text{Beta}(k, n(\beta+1) - k).$$

Note that setting $q = 0$ in (1) yields the expression for EV^p given by Siegel (1978). Taking $p = 0$ in (1) gives an equation for the factorial moments of G. This equation is equivalent to the expression for $P\{G = k\}$ given in Theorem 2.1 of Siegel and Holst (1982). To see this we expand the indicator of $\{G = k\}$ using the combinatorial identity

$$I_{\{G=k\}} = \sum_{j\geq k}(-1)^{j-k}\binom{j}{k}\binom{G}{j}.$$

Taking expectations on both sides of this identity and using (1) yields the desired expression for $P\{G = k\}$. The distribution of G was also obtained by Jewell and Romano (1982).

When $F(x) = x^{\beta}$, Theorem 3.1 of Siegel and Holst (1982) gives an explicit formula for $P\{G = k\}$ in terms of gamma functions. Result (1) can be used in the same way to obtain explicit formulas for the joint moments of V and G. Also note that combining result (2) and the formula of Siegel and Holst gives a complete description of the joint distribution of V and G when $F(x) = x^{\beta}$.

Proof of (1)

Choose points $X_1, X_2, \cdots, X_n$ uniformly and independently on a circle having unit circumference. Let $L_1, L_2, \cdots, L_n$ be i.i.d. from F. The jth random arc will be the arc having clockwise endpoint X_j and length L_j. For convenience we assume each arc is open and does not contain its own endpoints.

Following the basic argument of Robbins (1944) used by Siegel (1978) to prove his Theorem 4.1, we introduce p additional points $Y_1, Y_2, \cdots, Y_p$ chosen independently and uniformly on the circle. Define B to be the event that all of the points Y_i lie in the vacancy. It is clear that

$$V^p = P(B \mid X_1, \cdots, X_n, L_1, \cdots, L_n).$$

G is a function of $X_1, \cdots, X_n$, $L_1, \cdots, L_n$ so we have

$$\begin{aligned} E\binom{G}{q}V^p &= E\left[\binom{G}{q}E(I_B \mid X_1, \cdots, X_n, L_1, \cdots, L_n)\right] \\ &= E\binom{G}{q}I_B. \end{aligned} \tag{3}$$

Let A_i be the event that the point X_i is not covered by any arc. Now G can be expressed as

$$G = \sum_{j=1}^{n} I_{A_j}.$$

Let T be the collection of all $\binom{n}{q}$ subsets of $\{1, 2, \cdots, n\}$ having exactly q members. Elementary counting arguments yield

$$\binom{G}{q} = \sum_{\tau \in T} \prod_{j \in \tau} I_{A_j}.$$

Putting this expression in (3) and using the underlying exchangeability we obtain

$$E\binom{G}{q}V^p = \sum_{\tau\in\mathcal{T}} P\left(B\cap\left[\bigcap_{i\in\tau} A_i\right]\right) \tag{4}$$
$$= \binom{n}{q} P\left(B\cap\left[\bigcap_{i=1}^{q} A_i\right]\right).$$

Now we argue as in the proof of Theorem 2.1 in Siegel and Holst (1982). The event $B\cap[\bigcap_{i=1}^{q} A_i]$ occurs if none of the points $X_1,\cdots,X_q, Y_1,\cdots,Y_p$ are covered by the arcs. The points $X_1,\cdots,X_q, Y_1,\cdots,Y_p$ divide the circle into $p+q$ segments whose lengths will be denoted $S_1, S_2,\cdots,S_{p+q}$. More precisely, for $1\le i\le q$ take S_i to be the length of the segment having clockwise endpoint X_i. For $q+1\le i\le p+q$ take S_i to be the length of the segment having clockwise endpoint Y_{i-q}. The random variables $S_1, S_2,\cdots,S_{p+q}$ are exchangeable and their joint distribution is uniform on the simplex determined by $\sum_{i=1}^{p+q} S_i = 1$.

Let D be the event that none of the arcs $q+1, q+2,\cdots,n$ cover any of the points $X_1,\cdots,X_q, Y_1,\cdots,Y_p$. Then we can write

$$B\cap\left[\bigcap_{i=1}^{q} A_i\right] = D\cap\left[\bigcap_{i=1}^{q}\{L_i\le S_i\}\right].$$

Conditioning on $X_1, X_2,\cdots,X_q, Y_1, Y_2,\cdots,Y_p$ we find

$$P\left(\bigcap_{i=1}^{q}\{L_i\le S_i\} \mid X_1,\cdots,X_q,Y_1,\cdots,Y_p\right) = \prod_{i=1}^{q} F(S_i)$$

and

$$P(D\mid X_1,\cdots,X_q,Y_1,\cdots,Y_p) = \left[\sum_{j=1}^{p+q}\phi(S_j)\right]^{n-q}.$$

Using the conditional independence of $\bigcap_{i=1}^{q}\{L_i\le S_i\}$ and D we have

$$P\left(D\cap\left[\bigcap_{i=1}^{q}\{L_i\le S_i\}\right]\right) = E\left[\prod_{i=1}^{q} F(S_i)\right]\left[\sum_{j=1}^{p+q}\phi(S_j)\right]^{n-q}.$$

For further details see Siegel and Holst (1982). Substituting this expression in (4) completes the proof.

Proof of (2)

Again let the ith arc have clockwise endpoint X_i and length L_i. For now, the arc lengths L_i will have an arbitrary distribution F on $(0,1)$. Let W_i denote the counterclockwise endpoint of the ith arc and let Z_i be the length of the uncovered gap which begins at W_i. If W_i is covered, we set $Z_i = 0$. The random variables $Z_1, Z_2, \cdots, Z_n$ are exchangeable, $Z_i > 0$ for exactly G values of i, and $\Sigma_i Z_i = V$.

We shall now obtain an expression for $P(Z_1 > t_1, Z_2 > t_2, \cdots, Z_m > t_m)$ where $m \leq n$ and $t_i > 0$ for all i. The points $X_1, X_2, \cdots, X_m$ divide the circle into m segments whose lengths will be denoted $S_1, S_2, \cdots, S_m$. Take S_i to be the length of the segment having clockwise endpoint X_i. Let D be the event that none of the random arcs numbered $m+1, m+2, \cdots, n$ intersect the m intervals on the circle having lengths $t_1, t_2, \cdots, t_m$ and clockwise endpoints $W_1, W_2, \cdots, W_m$ respectively. Then

$$\bigcap_{i=1}^{m}\{Z_i > t_i\} = \left[\bigcap_{i=1}^{m}\{S_i > L_i + t_i\}\right] \cap D.$$

Conditioning on the locations of $X_1, X_2, \cdots, X_m$ and duplicating the argument used by Siegel and Holst (1982) to prove their theorem 2.1 leads to

$$(5) \qquad P\left(\bigcap_{i=1}^{m}\{Z_i > t_i\}\right) = E\left\{\left[\prod_{i=1}^{m} F(S_i - t_i)\right]\left[\sum_{j=1}^{m} \phi(S_j - t_j)\right]^{n-m}\right\}$$

where $S_1, S_2, \cdots, S_m$ are the lengths of the spaces between m points chosen uniformly and independently on a circle with unit circumference. ϕ is as given in (1). Note that $F(u) = \phi(u) = 0$ for $u < 0$.

This expression can be somewhat simplified. It is well known (see Chapter 1, Problem 23 of Feller (1971)) that

$$(6) \qquad P(S_1 > u_1, S_2 > u_2, \cdots, S_m > u_m) = \left(1 - \sum_{i=1}^{m} u_i\right)_+^{m-1}$$

when $u_i \geq 0$ for all i. Define $C = \bigcap_{i=1}^{m}\{S_i > t_i\}$ and $T = \sum_{i=1}^{m} t_i$. Using (6) it is easy to find the conditional distribution of $S_1, \cdots, S_m$ given C;

$$(7) \qquad \mathcal{L}(S_1 - t_1, S_2 - t_2, \cdots, S_m - t_m \mid C) = \mathcal{L}((1-T)(S_1, S_2, \cdots, S_m)).$$

From (5), (6) and (7) we obtain

$$
(8)\qquad
\begin{aligned}
P\left(\bigcap_{i=1}^{m}\{Z_i > t_i\}\right) &= P(C)E\left\{\left[\prod_{i=1}^{m} F(S_i - t_i)\right]\left[\sum_{j=1}^{m}\phi(S_j - t_j)\right]^{n-m} \mid C\right\} \\
&= (1-T)_+^{m-1} E\left\{\left[\prod_{i=1}^{m} F((1-T)S_i)\right]\left[\sum_{j=1}^{m}\phi((1-T)S_j)\right]^{n-m}\right\}.
\end{aligned}
$$

When $F(x) = x^{\beta}$ so that $\phi(x) = x^{\beta+1}/\beta + 1$, the factors involving $(1-T)$ can be taken outside the expectation so that (8) becomes

$$
(9)\qquad
\begin{aligned}
P\left(\bigcap_{i=1}^{m}\{Z_i > t_i\}\right) &= (1-T)_+^{n(\beta+1)-1} E\left\{\left[\prod_{i=1}^{m} S_i^{\beta}\right]\left[\sum_{j=1}^{m} S_j^{\beta+1}/\beta + 1\right]^{n-m}\right\} \\
&= (1-T)_+^{n(\beta+1)-1} P\left(\bigcap_{i=1}^{m}\{Z_i > 0\}\right)
\end{aligned}
$$

Define $B_m = \bigcap_{i=1}^{m}\{Z_i > 0\}$. Let $f_m(t_1, t_2, \cdots, t_m)$ denote the joint density of the conditional distribution of $Z_1, Z_2, \cdots, Z_m$ given B_m. By partial differentiation of (9) we find that

$$
(10)\qquad f_m(t_1, t_2, \cdots, t_m) = \left(1 - \sum_{i=1}^{m} t_i\right)_+^{n(\beta+1)-m-1}
$$

which is the density of a Dirichlet distribution. For $1 \le k \le n$ define $Y_k = \sum_{i=1}^{k} Z_i$ so that $V = Y_n$. Using (10) and a standard property of Dirichlet distributions we see that for $k \le m$

$$
\mathcal{L}(Y_k \mid B_m) = \text{Beta}(k, n(\beta+1) - k)
$$

which has a density proportional to $x^{k-1}(1-x)^{n(\beta+1)-k-1}$. Note that this conditional distribution does not depend on m. For convenience, define $F_k(t) = P(Y_k \le t \mid B_m)$ for $k \le m$.

To complete the proof we modify an argument due to Holst (1983). Let I_j be the indicator

of the event $\{Z_j > 0\}$. Using the exchangeability of $Z_1, Z_2, \cdots, Z_n$ we have

$$\begin{aligned} P(V \le t \text{ and } G = k) &= \binom{n}{k} E I_{\{Y_k \le t\}} \prod_{i=1}^{k} I_i \prod_{i=k+1}^{n} (1 - I_i) \\ &= \binom{n}{k} \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} E I_{\{Y_k \le t\}} \prod_{j=1}^{k+i} I_j \\ &= \binom{n}{k} \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} P(Y_k \le t \mid B_{k+i}) P(B_{k+i}) \\ &= F_k(t) \left(\binom{n}{k} \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} E \prod_{j=1}^{k+i} I_j \right) \\ &= F_k(t) P(G = k). \end{aligned}$$

Therefore $P(V \le t \mid G = k) = F_k(t)$ and the conditional distribution of V has the desired Beta distribution.

Acknowledgment

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Random arcs having random sizes are placed on a circle. Let V be the length of the uncovered portion of the circle and G be the number of uncovered gaps on the circle. Results are presented concerning the joint moments of V and G and the conditional distribution of V given G.

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